Mode Theory of Lossless Periodically Distributed Parametric Amplifiers*

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Summary—In this paper, an operator $T\theta$ is introduced for the analysis of the periodically distributed parametric amplifier. The operator is the product of a diagonal matrix expressing the pumping phase relation and the T matrix of the basic section of the amplifier. The eigenvectors of $T\theta$ are called the "modes" of the amplifier. The orthogonality properties of the modes are proved in a similar way as for the conventional mode theory. Finally, an expression is derived for the power gain of the amplifier as an application of the theory.

I. Introduction

CONSIDERABLE attention has been given recently to the parametric amplifier mainly because of the possibility of low-noise characteristics. The limitation of bandwidth[1] has been removed by the proposal of the traveling wave parametric amplifier; this proposal has been made by Miyakawa[2] and by Tien and Suhl[3] independently. The loss of available ferrites, however, requires a large amount of pumping power for the traveling wave ferromagnetic amplifier. In this regard, the traveling wave parametric amplifier with semiconductor diodes, as the active elements periodically loaded in the transmission line is more promising. As a matter of fact, some successful results already have been reported.[4] The term "periodically distributed parametric amplifier" will be used in this paper for the amplifier of this type to distinguish it from the one with uniformly distributed variable reactances. The theoretical study of the periodically distributed parametric amplifier was first undertaken by Saito. It is shown that the growing and decreasing waves can propagate in the lossless transmission line periodically loaded with the variable capacitors, of which the invariant parts are effectively cancelled out. These growing and decreasing waves are, naturally, very similar to those of the traveling wave amplifier discussed by Miyakawa, Tien, and Suhl. The extension of Saito's work leads to the eigenvalue problem of an operator $T\theta$, the product of a diagonal matrix expressing the pumping phase relation, and the T matrix of the basic section of the amplifier. The eigenvectors of the operator $T\theta$ may be called the modes of the periodically distributed amplifier. Presentation of the theory of these modes is the aim of this paper. The orthogonality relations between the modes are proved in a similar way as for the conventional mode theory.[5] Finally, the first approximation of the gain of the amplifier is derived as an application of the theory.

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[1] H. Heffner and G. Wade, "Gain, bandwidth, and noise characteristics of the variable-parameter amplifier," *J. Appl. Phys.*, vol. 29, pp. 1321–1331; September, 1958.

[2] H. Miyakawa, "Amplification and frequency conversion in propagating circuits," *Inst. Elec. Comm. Engrs., Japan Nat'l. Convention Record*, p. 8; November, 1957 (in Japanese).

[3] P. K. Tien and H. Suhl, "A traveling wave ferromagnetic amplifier," Proc. IRE, vol. 46, pp. 700–706; April, 1958.

[4] R. S. Engelbrecht, "A low-noise nonlinear reactance traveling wave amplifier," Proc. IRE, vol. 46, p. 1655; September, 1958.

II. Introduction of the Operator $T\theta$

For the sake of simplicity, we shall consider the lossless two terminal pair networks with a variable capacitor as illustrated in Fig. 1. Fig. 1 (a) and (b) are identical two-terminal pair networks. The invariant part of the variable capacitor is divided into two parts, each of which is included in (*a*) and (*b*). Z_0 is the image impedance of (*a*) or (*b*) looking into the outside terminal, and Z_0' is the impedance looking into the inside terminal. (The prime notation indicates the value of the inside terminals.) In this section we shall indicate whether a quantity refers to the angular frequency ω_1 or ω_2 by the last subscript 1 or 2, respectively. We often omit this last subscript if the equation holds for both frequencies.

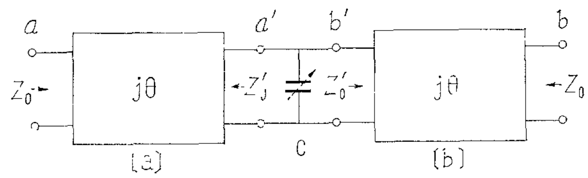

Fig. 1—Basic section of the amplifier.

The voltage and current at each terminal in Fig. 1 are, in terms of the incident waves (subscript i) and the reflected waves (subscript r),

$$V_a = \sqrt{Z_0}(a_i + a_r) \qquad V_a' = \sqrt{Z_0'}(a_i e^{-j\theta} + a_r e^{j\theta})$$

$$I_a = \frac{1}{\sqrt{Z_0}}(a_i - a_r) \qquad I_a' = \frac{1}{\sqrt{Z_0'}}(a_i e^{-j\theta} - a_r e^{j\theta}) \qquad (1)$$

$$V_b = \sqrt{Z_0}\,(b_i + b_r) \qquad V_b' = \sqrt{Z_0'}(b_i e^{j\theta} + b_r e^{-j\theta})$$

$$I_b = \frac{1}{\sqrt{Z_0}}(b_i - b_r) \qquad I_b' = \frac{1}{\sqrt{Z_0'}}(b_i e^{j\theta} - b_r e^{-j\theta}). \qquad (2)$$

[5] H. A. Haus, "Coupling of modes of propagation," M.I.T. Rep. (unpublished).

Since the voltages at the c-terminals must be equal,

$$V_a' = V_b'.$$

For ω_1, the above equation becomes

$$a_{i1}e^{-j\theta_1} + a_{r1}e^{j\theta_1} = b_{i1}e^{j\theta_1} + b_{r1}e^{-j\theta_1}. \tag{3}$$

The equation of continuity is

$$I_a' = I_b' + I_c \tag{4}$$

where I_c is the current through C. I_c is related to the voltage across C. If the pumping angular frequency ω_p is equal to the sum of ω_1 and ω_2, that is, if

$$\omega_1 + \omega_2 = \omega_p, \tag{5}$$

the relation is[6]

$$I_c = \begin{pmatrix} I_1 \\ I_2^* \end{pmatrix} = \begin{pmatrix} 0 & j\omega_1 \dfrac{c}{2} \\ -j\omega_2 \dfrac{c^*}{2} & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2^* \end{pmatrix} \tag{6}$$

where the asterisk denotes the complex conjugate. Using (1), (2), (3), and (6), we rewrite (4) in the form

$$a_{i1}e^{-j\theta_1} - a_{r1}e^{j\theta_1} - j\omega_1 \frac{c}{2} \sqrt{Z_{01}'Z_{02}'^*}(a_{i2}^*e^{j\theta_2} + a_{r2}^*e^{-j\theta_2}) = b_{i1}e^{j\theta_1} - b_{r1}e^{-j\theta_1}. \tag{7}$$

From (3) and (7), we have

$$b_{i1} = a_{i1}e^{-2j\theta_1} - j\omega_1 \frac{c}{4} \sqrt{Z_{01}'Z_{02}'^*}(a_{i2}^*e^{j(\theta_2-\theta_1)} + a_{r2}^*e^{-j(\theta_1+\theta_2)}). \tag{8}$$

Similarly, all the b's can be expressed in terms of the a's. The result is, in the matrix form,

$$B = TA \tag{9}$$

where

$$T = \left(\begin{array}{cc|cc} e^{-2j\theta_1} & 0 & -j\omega_1 c \dfrac{\sqrt{Z_{01}'Z_{02}'^*}}{4} e^{j(\theta_2-\theta_1)} & -j\omega_1 c \dfrac{\sqrt{Z_{01}'Z_{02}'^*}}{4} e^{-j(\theta_1+\theta_2)} \\ 0 & e^{2j\theta_1} & j\omega_1 c \dfrac{\sqrt{Z_{01}'Z_{02}'^*}}{4} e^{j(\theta_1+\theta_2)}, & j\omega_1 c \dfrac{\sqrt{Z_{01}'Z_{02}'^*}}{4} e^{j(\theta_1-\theta_2)} \\ \hline j\omega_2 c^* \dfrac{\sqrt{Z_{01}'Z_{02}'^*}}{4} e^{j(\theta_2-\theta_1)}, & j\omega_2 c^* \dfrac{\sqrt{Z_{01}'Z_{02}'^*}}{4} e^{j(\theta_1+\theta_2)} & e^{2j\theta_2} & 0 \\ -j\omega_2 c^* \dfrac{\sqrt{Z_{01}'Z_{02}'^*}}{4} e^{-j(\theta_1+\theta_2)}, & -j\omega_2 c^* \dfrac{\sqrt{Z_{01}'Z_{02}'^*}}{4} e^{j(\theta_1-\theta_2)} & 0 & e^{-2j\theta_2} \end{array}\right)$$

$$A = \begin{pmatrix} a_{i1} \\ a_{r1} \\ \hline a_{i2}^* \\ a_{r2}^* \end{pmatrix} \qquad B = \begin{pmatrix} b_{i1} \\ b_{r1} \\ \hline b_{i2}^* \\ b_{r2}^* \end{pmatrix}. \tag{10}$$

The vector A expresses the waves at the input and the vector B at the output of the basic circuit. The circuit in Fig. 1 is represented by the square matrix T, which transforms A into B.

Next we shall consider the n similar circuits connected in cascade. The variable capacitor of each circuit has the pumping phase lagged by $2\theta_p$ from that of the preceding one. These circuits are represented by the similar matrices to T, but they have $ce^{-2j\theta_p}$, $ce^{-4j\theta_p}$, $\cdots$ $ce^{-2(n-1)j\theta_p}$ in place of c.

For the analysis of the cascade connections of the same circuits, it is well known that the solutions of the eigenvalue problem of T are of great help: the circuits as a whole transform each eigenvector to the same eigenvector multiplied by (the eigenvalue)^n. The circuits under consideration are, however, different from each other, and the solutions of the eigenvalue problem of T are of no advantages at all.

Here we assume that T transforms the ω_1 components a_1 and the ω_2 components a_2 of A to $\gamma_1 a_1$ and $\gamma_2 a_2$, respectively, where γ_1 and γ_2 are scalers.

If we write T in the form

$$T = \left(\begin{array}{c|c} t_1 & cm_1 \\ \hline c^*m_2 & t_2 \end{array}\right), \tag{11}$$

(9) becomes

$$\begin{pmatrix} b_1 \\ \hline b_2^* \end{pmatrix} = \left(\begin{array}{c|c} t_1 & cm_1 \\ \hline c^*m_2 & t_2 \end{array}\right) \begin{pmatrix} a_1 \\ \hline a_2^* \end{pmatrix} = \begin{pmatrix} \gamma_1 a_1 \\ \hline \gamma_2^* a_2^* \end{pmatrix}. \tag{12}$$

The operator of the second section is

$$\left(\begin{array}{c|c} t_1 & ce^{-2j\theta_p}m_1 \\ \hline c^*e^{2j\theta_p}m_2 & t_2 \end{array}\right).$$

[6] H. E. Rowe, "Some general properties of nonlinear elements. II. Small signal theory," Proc. IRE, vol. 46, pp. 850–860; May, 1958.

From (12), we have

$$\left(\begin{array}{c|c} t_1 & ce^{-2j\theta_p}m_1 \\ \hline c^*e^{2j\theta_p}m_2 & t_2 \end{array}\right)\left(\begin{array}{c} a_1 \\ \hline a_2^*e^{2j\theta_p} \end{array}\right) = \left(\begin{array}{c} \gamma_1 a_1 \\ \hline \gamma_2^* a_2^* e^{2j\theta_p} \end{array}\right). \quad (13)$$

Hence, if we assume the relevance

$$\gamma_2^* = \gamma_1 e^{2j\theta_p}, \quad (14)$$

the output of the second section becomes

$$\left(\begin{array}{c|c} t_1 & ce^{-2j\theta_p}m_1 \\ \hline c^*e^{2j\theta_p}m_2 & t_2 \end{array}\right)\left(\begin{array}{c|c} t_1 & cm_1 \\ \hline c^*m_2 & t_2 \end{array}\right)\left(\begin{array}{c} a_1 \\ \hline a_2^* \end{array}\right)$$

$$= \left(\begin{array}{c|c} t_1 & ce^{-2j\theta_p}m_1 \\ \hline c^*e^{2j\theta_p}m_2 & t_2 \end{array}\right)\left(\begin{array}{c} \gamma_1 a_1 \\ \hline \gamma_2^* a_2^* \end{array}\right) = \left(\begin{array}{c} \gamma_1^2 a_1 \\ \hline \gamma_2^{*2} a_2^* \end{array}\right). \quad (15)$$

Similarly, the output of the nth section is

$$\left(\begin{array}{c|c} t_1 & ce^{-2(n-1)j\theta_p}m_1 \\ \hline c^*e^{2(n-1)j\theta_p}m_2 & t_2 \end{array}\right)\cdots$$

$$\cdot\left(\begin{array}{c|c} t_1 & ce^{-2j\theta_p}m_1 \\ \hline c^*e^{2j\theta_p}m_2 & t_2 \end{array}\right)\left(\begin{array}{c|c} t_1 & cm_1 \\ \hline c^*m_2 & t_2 \end{array}\right)\left(\begin{array}{c} a_1 \\ \hline a_2^* \end{array}\right)$$

$$= \left(\begin{array}{c} \gamma_1^n a_1 \\ \hline \gamma_2^{*n} a_2^* \end{array}\right). \quad (16)$$

This is a very simple relation. Thus we have shown that the solutions of (12) may play an important part in our analysis.

If we put

$$\gamma_1 = \lambda e^{-j\theta_p}, \qquad \gamma_2^* = \lambda e^{j\theta_p}, \quad (17)$$

then (14) is satisfied. We now rewrite (12) in the form

$$(T - \lambda I_\theta)A = 0 \quad (18)$$

where

$$I_\theta = \begin{bmatrix} e^{-j\theta_p} & 0 & 0 & 0 \\ 0 & e^{-j\theta_p} & 0 & 0 \\ 0 & 0 & e^{j\theta_p} & 0 \\ 0 & 0 & 0 & e^{j\theta_p} \end{bmatrix}. \quad (19)$$

The vector A satisfying (18) is transformed into $\lambda^n I_\theta^n A$ by the transformation of the left hand side of (16). Multiplying (18) by I_θ^{-1} from the left, we obtain

$$(T_\theta - \lambda I)A = 0 \quad (20)$$

where I is the unit matrix and

$$T_\theta = T_\theta^{-1}T = I_\theta^* T. \quad (21)$$

Eq. (20) has just the conventional form of the eigenvalue problems. As is well known, there are four independent eigenvectors (m eigenvectors in case of m dimensional space) and an arbitrary vector can be expressed as a linear combination of them. Each eigenvector A_k is independently transformed by the amplifier into $\lambda_k^n I_\theta^n A_k$, where λ_k is the eigenvalue of the eigenvector A_k. For this reason, the eigenvectors of T_θ may be called the modes of the periodically distributed parametric amplifier.

III. The Orthogonality Properties of the Modes

The eigenvectors of T_θ have certain properties of orthogonality which are important when we wish to express a vector as the sum of the eigenvectors. The orthogonality theorems take, of course, different forms from the conventional circuits. The theorems hold in a more general case than the particular amplifier discussed in Section II. We shall prove them in the general case, using one of the Manley-Rowe relations.[7]

For the lossless parametric circuit with ω_p satisfying (5), the Manley-Rowe relation is

$$\frac{W_1}{\omega_1} - \frac{W_2}{\omega_2} = 0 \quad (22)$$

where W_1 and W_2 represent the real powers flowing into the circuit at the angular frequencies ω_1 and ω_2, respectively.

If ω_1 and ω_2 are both in the pass-band of the two-terminal pair network, using (1) and (2), from (22) we obtain

$$\frac{1}{\omega_1}\mathrm{Re}\,(V_{a1}I_{a1}^* - V_{b1}I_{b1}^*) - \frac{1}{\omega_2}\mathrm{Re}\,(V_{a2}I_{a2}^* - V_{b2}I_{b2}^*)$$

$$= \frac{1}{\omega_1}(|a_{i1}|^2 - |a_{r1}|^2 - |b_{i1}|^2 + |b_{r1}|^2)$$

$$- \frac{1}{\omega_2}(|a_{i2}|^2 - |a_{r2}|^2 - |b_{i2}|^2 + |b_{r2}|^2)$$

$$= A^+\Omega^{-1}A - B^+\Omega^{-1}B$$

$$= A^+(\Omega^{-1} - T^+\Omega^{-1}T)A = 0 \quad (23)$$

where the symbol $^+$ denotes the complex conjugate transposed matrix and

$$\Omega^{-1} = \begin{bmatrix} \frac{1}{\omega_1} & 0 & 0 & 0 \\ 0 & -\frac{1}{\omega_1} & 0 & 0 \\ 0 & 0 & -\frac{1}{\omega_2} & 0 \\ 0 & 0 & 0 & \frac{1}{\omega_2} \end{bmatrix}. \quad (24)$$

Since (23) must hold for every A,

$$\Omega^{-1} = T^+\Omega^{-1}T. \quad (25)$$

This is the condition which T of all the parametric circuits should satisfy. (See Appendix, Section A.)

[7] J. M. Manley and H. E. Rowe, "Some general properties of nonlinear elements—part I. General energy relations," Proc. IRE, vol. 44, pp. 904–913; July, 1956.

If two T's, T_a and T_b, satisfy (25), then

$$(T_aT_b)^+\Omega^{-1}(T_aT_b) = T_b^+T_a^+\Omega^{-1}T_aT_b = T_b^+\Omega^{-1}T_b = \Omega^{-1}. \quad (26)$$

Eq. (26) shows that the product of two T's, (T_aT_b), again satisfies (25). It is worth noting that the unit matrices I and T of the conventional circuits also satisfy (25). Since I_θ^{-1} satisfies (25), T_θ also satisfies (25):

$$\Omega^{-1} = T_\theta^+\Omega^{-1}T_\theta. \quad (27)$$

If λ_k is an eigenvalue of T_θ, the determinant of $(T_\theta - \lambda_k I)$ vanishes:

$$\det (T_\theta - \lambda_k I) = 0. \quad (28)$$

Taking the complex conjugate transpose of (28), we have

$$\det (T_\theta^+ - \lambda_k^* I) = 0.$$

Since $\det (T_\theta) \neq 0$, $\lambda_k \neq 0$. From these relations, we have

$$\begin{aligned} \det (T_\theta^+ - \lambda_k^* I) \det (\Omega^{-1}T_\theta) &= \det (T_\theta^+\Omega^{-1}T_\theta - \lambda_k^*\Omega^{-1}T_\theta) \\ &= \det (\Omega^{-1} - \lambda_k^*\Omega^{-1}T_\theta) \\ &= \det (\Omega^{-1}\lambda_k^*) \det \left(\frac{1}{\lambda_k^*} I - T_\theta\right) = 0. \end{aligned}$$

The final result is

$$\det \left(T_\theta - \frac{1}{\lambda_k^*} I\right) = 0. \quad (29)$$

It says, if λ_k is an eigenvalue then $1/\lambda_k^*$ is also an eigenvalue of T_θ. In other words, when $|\lambda_k| \neq 1$, the eigenvalues λ_k and $1/\lambda_k^*$ appear always in pairs. When $|\lambda_k| = 1$, $1/\lambda_k^*$ is equal to λ_k, and the result is trivial.

If λ_k and λ_l are the two eigenvalues of T_θ,

$$T_\theta A_k = \lambda_k A_k. \quad (30)$$

$$T_\theta A_l = \lambda_l A_l. \quad (31)$$

From (31), we have

$$\frac{1}{\lambda_l^*} (T_\theta A_l)^+ = \frac{1}{\lambda_l^*} A_l^+T_\theta^+ = A_l^+.$$

Multiplying by $\Omega^{-1}T_\theta A_k$ from the right and using (27), we have

$$\frac{1}{\lambda_l^*} A_l^+\Omega^{-1}A_k = A_l^+\Omega^{-1}T_\theta A_k.$$

Multiplying (30) by $A_l^+\Omega^{-1}$ from the left and substituting in the above equation, we obtain

$$\left(\lambda_k - \frac{1}{\lambda_l^*}\right) A_l^+\Omega^{-1} A_k = 0. \quad (32)$$

If $\lambda_k \neq 1/\lambda_l^*$, from (32), we have

$$A_l^+\Omega^{-1}A_k = 0. \quad (33)$$

In case $|\lambda_k| \neq 1$, since $\lambda_k \neq 1/\lambda_k^*$, we can set $l = k$ in (33); that is,

$$A_k^+\Omega^{-1}A_k = 0 \qquad (|\lambda_k| \neq 1). \quad (34)$$

Next, we expand ΩA_k in terms of the modes in the form

$$\Omega A_k = \sum_j \alpha_j A_j.$$

Multiplying by $A_k^+\Omega^{-1}$ from the left, we have

$$A_k^+A_k = \sum_j \alpha_j A_k^+\Omega^{-1}A_j \neq 0. \quad (35)$$

Assuming that λ_k is not degenerate and using (33), when $|\lambda_k| = 1$, we obtain

$$A_k^+\Omega^{-1}A_k \neq 0 \qquad (|\lambda_k| = 1). \quad (36)$$

If $|\lambda_k| \neq 1$, there is always the eigenvector A_l corresponding to the eigenvalue $1/\lambda_k^*$. In this case, (35) becomes

$$A_k^+\Omega^{-1}A_l \neq 0$$

which can be rewritten in the form

$$A_l^+\Omega^{-1}A_k \neq 0. \quad (37)$$

Here, we define $\tilde{A}_k$ by

$$\tilde{A}_k = A_k^+ \qquad \text{if } |\lambda_k| = 1 \quad (38)$$

$$\tilde{A}_k = A_l^+ \qquad \text{if } |\lambda_k| \neq 1 \quad (39)$$

where A_l is the eigenvector corresponding to the eigenvalue $1/\lambda_k^*$. Then, (33), (34), (36), and (37) become

$$\begin{aligned} \tilde{A}_l\Omega^{-1}A_k &= 0 \qquad (l \neq k) \\ \tilde{A}_k\Omega^{-1}A_k &\neq 0. \end{aligned} \quad (40)$$

These are the orthogonality theorems which we wished to prove.

In the case of degeneracy, the above proof does not necessarily hold. It is, however, always possible to introduce the eigenvectors in such a way as to secure the orthogonality, and we are justified in assuming (40) even in case of degeneracy. (See Appendix, Section B.)

IV. Power Gain of the Amplifier

In this section, we shall derive an expression for the power gain of the periodically distributed parametric amplifier.

We need the solutions of the eigenvalue problem (20). In the preceding sections, we have imposed no conditions on θ_p. Here we shall confine ourselves to the case of synchronous pumping:

$$\theta_p = \theta_1 + \theta_2. \quad (41)$$

All the eigenvalues and the corresponding eigenvectors can be obtained by the standard method of algebra, or by the method of perturbation. To the first order of approximation, they are

$$\lambda_1 = (1+\delta)e^{j(\theta_2-\theta_1)}$$

$$A_1 = \begin{pmatrix} 1 \\ -j\dfrac{\delta}{2\sin 2\theta_1} \\ j\sqrt{\dfrac{\omega_2}{\omega_1}}\dfrac{c^*}{|c|}e^{-j\theta_p} \\ -\sqrt{\dfrac{\omega_2}{\omega_1}}\dfrac{c^*}{|c|}\dfrac{\delta e^{-j\theta_p}}{2\sin 2\theta_2} \end{pmatrix}$$

$$\lambda_2 = e^{j(3\theta_1+\theta_2)}$$

$$A_2 = \begin{pmatrix} 0 \\ 1 \\ \sqrt{\dfrac{\omega_2}{\omega_1}}\dfrac{c^*}{|c|}\dfrac{\delta e^{-j\theta_p}}{2\sin 2\theta_1} \\ -\sqrt{\dfrac{\omega_2}{\omega_1}}\dfrac{c^*}{|c|}\dfrac{\delta e^{-j\theta_p}}{2\sin 2\theta_p} \end{pmatrix}$$

$$\lambda_3 = (1-\delta)e^{j(\theta_2-\theta_1)}$$

$$A_3 = \begin{pmatrix} 1 \\ j\dfrac{\delta}{2\sin 2\theta_1} \\ -j\sqrt{\dfrac{\omega_2}{\omega_1}}\dfrac{c^*}{|c|}e^{-j\theta_p} \\ -\sqrt{\dfrac{\omega_2}{\omega_1}}\dfrac{c^*}{|c|}\dfrac{\delta e^{-j\theta_p}}{2\sin 2\theta_2} \end{pmatrix}$$

$$\lambda_4 = e^{-j(\theta_1+3\theta_2)}$$

$$A_4 = \begin{pmatrix} \sqrt{\dfrac{\omega_1}{\omega_2}}\dfrac{c}{|c|}\dfrac{\delta e^{j\theta_p}}{2\sin 2\theta_2} \\ -\sqrt{\dfrac{\omega_1}{\omega_2}}\dfrac{c}{|c|}\dfrac{\delta e^{j\theta_p}}{2\sin 2\theta_p} \\ 0 \\ 1 \end{pmatrix} \qquad (42)$$

where

$$\delta = \sqrt{\omega_1\omega_2}\,|c|\,\frac{\sqrt{Z_{01}'Z_{02}'^*}}{4}. \qquad (43)$$

If ω_1 and ω_2 are in the pass-band, as we have assumed, δ is real and we have $\lambda_1 = 1/\lambda_3^*$ which we proved in Section III. A_1 and A_3 represent the growing and decreasing waves, for $|\lambda_1|$ is greater than unity and $|\lambda_3|$ is smaller than unity. It is worth noting that they are almost the incident waves. A_2 and A_4 are the reflected waves, of which the propagation constants do not change in this approximation. The orthogonality theorems (40) are satisfied to the same order of approximation.

For the calculation of the power gain, we first express the input vector A as the sum of the eigenvectors in the form

$$A = \sum_k \alpha_k A_k. \qquad (44)$$

A_k is transformed into $\lambda_k^n I_\theta^n A_k$ by the amplifier. Hence, for the output vector B, we have

$$B = \sum_k \alpha_k \lambda_k^n \mathrm{I}_\theta^n A_k. \qquad (45)$$

Multiplying by $\tilde{A}_k\Omega^{-1}I_\theta^{-n}$ from the left, because of the orthogonality properties of the modes, we obtain

$$\tilde{A}_k\Omega^{-1}\mathrm{I}_\theta^{-n}B = \alpha_k\lambda_k^n\tilde{A}_k\Omega^{-1}A_k.$$

Therefore

$$\alpha_k = \frac{\tilde{A}_k\Omega^{-1}I_\theta^{-n}B}{\lambda_k^n\tilde{A}_k\Omega^{-1}A_k}. \qquad (46)$$

For simplicity, we assume that the output is terminated with Z_0: $b_{r1} = b_{r2}^* = 0$. Then, from (42), (44), and (46), we have

$$A = \begin{pmatrix} \dfrac{1}{2e^{-jn(\theta_1-\theta_2)}}\left(b_{i1}e^{jn\theta_p}\left\{\dfrac{1}{(1+\delta)^n}+\dfrac{1}{(1-\delta)^n}\right\} \underbrace{-j\sqrt{\dfrac{\omega_1}{\omega_2}}\dfrac{c}{|c|}b_{i2}^*e^{-j(n-1)\theta_p}\left\{\dfrac{1}{(1+\delta)^n}-\dfrac{1}{(1-\delta)^n}\right\}}_{\text{the order of }\delta}\right) \\ j\sqrt{\dfrac{\omega_2}{\omega_1}}\dfrac{c^*}{|c|}\dfrac{e^{-j\theta_p}}{2e^{-jn(\theta_1-\theta_2)}}\left(\underbrace{b_{i1}e^{jn\theta_p}\left\{\dfrac{1}{(1+\delta)^n}-\dfrac{1}{(1-\delta^n)}\right\}}_{\text{the order of }\delta} -j\sqrt{\dfrac{\omega_1}{\omega_2}}\dfrac{c}{|c|}b_{i2}^*e^{-j(n-1)\theta_p}\left\{\dfrac{1}{(1+\delta)^n}+\dfrac{1}{(1-\delta)^n}\right\}\right) \end{pmatrix}. \qquad (47)$$

We further assume that the input is also terminated with Z_{02} at ω_2:

$$V_2 = -Z_{02}I_2.$$

This means

$$a_{i2} = 0. \qquad (48)$$

From (47) and (48), we have

$$b_{i1}e^{jn\theta_p}\left\{\frac{1}{(1+\delta)^n}-\frac{1}{(1-\delta)^n}\right\}$$

$$= j\sqrt{\frac{\omega_1}{\omega_2}}\frac{c}{|c|}b_{i2}^*e^{-j(n-1)\theta_p}\left\{\frac{1}{(1+\delta)^n}+\frac{1}{(1-\delta)^n}\right\}. \qquad (49)$$

Substituting in a_{i1} in (47), we obtain

$$a_{i1} = \frac{1}{e^{-jn(\theta_1-\theta_2)}} b_{i1} e^{jn\theta_p} \left\{ \frac{2}{(1+\delta)^n + (1-\delta)^n} \right\}. \tag{50}$$

The power gain is the ratio of the output power $|b_{i1}|^2$ to the input power $|a_{i1}|^2$. Thus we find

$$G = \frac{|b_{i1}|^2}{|a_{i1}|^2} = \left\{ \frac{(1+\delta)^n + (1-\delta)^n}{2} \right\}^2 = \cosh^2 n\delta. \tag{51}$$[8]

We took $|a_{i1}|^2$ as the input power instead of $|a_{i1}|^2 - |a_{r1}|^2$. The reason for this choice is that the net input power $|a_{i1}|^2 - |a_{r1}|^2$ may become negative because, from (47), a_{r1} is of the order of δ and a_{i1} becomes of the order of δ if G is of the order of $1/\delta^2$. In this case a circulator can be employed to secure the stability.

Appendix

A. The Form of Ω^{-1}

In case ω_2 is in the stop-band (Z_{02} is pure imaginary) while ω_1 remains in the pass-band, the same manipulation as (23) leads to

$$\Omega^{-1} = \begin{bmatrix} \frac{1}{\omega_1} & 0 & 0 & 0 \\ 0 & -\frac{1}{\omega_1} & 0 & 0 \\ 0 & 0 & 0 & \pm j\frac{1}{\omega_2} \\ 0 & 0 & \mp j\frac{1}{\omega_2} & 0 \end{bmatrix} \tag{52}$$

where the upper signs in the matrix refer to the inductive Z_{02} and the lower signs to the capacitive Z_{02}. With this Ω^{-1} in place of (24), the orthogonality theorems can be proved without alteration.

B. The Orthogonality in the Case of Degeneracy

We shall consider the case of double degeneracy. Let A_1 and A_2 be the independent degenerate eigenvectors with $|\lambda_1| \neq 1$. Because of the degeneracy, we have

$$\frac{d}{d\lambda_1} \det (T_\theta - \lambda_1 I) = 0. \tag{53}$$

In a similar way as for (29), we obtain

$$\frac{d}{d\left(\frac{1}{\lambda^*_1}\right)} \det \left(T_\theta - \frac{1}{\lambda^*_1} I \right) = 0 \tag{54}$$

proving that $1/\lambda_1^*$ is also two fold. We denote the two independent eigenvectors corresponding to the eigenvalue $1/\lambda_1^*$ by A_{-1} and A_{-2}. In terms of the modes, ΩA_1 and ΩA_2 are

$$\Omega A_1 = \sum_k \alpha_k A_k$$
$$\Omega A_2 = \sum_k \beta_k A_k. \tag{55}$$

If we put

$$A_a = A_1 + aA_2$$
$$A_b = A_1 - aA_2, \tag{56}$$

then, using (33) and (55), we have

$$\begin{aligned} 0 \neq A_a^+ A_a &= A_a^+(A_1 + aA_2) \\ &= A_a^+ \Omega^{-1}(\alpha_{-1}A_{-1} + \alpha_{-2}A_{-2} + a\beta_{-1}A_{-1} + a\beta_{-2}A_{-2}) \\ 0 \neq A_b^+ A_b &= A_b^+(A_1 - aA_2) \\ &= A_b^+ \Omega^{-1}(\alpha_{-1}A_{-1} + \alpha_{-2}A_{-2} - a\beta_{-1}A_{-1} - a\beta_{-2}A_{-2}). \end{aligned}$$

Hence, if we define A_{-a} and A_{-b} by

$$\begin{aligned} A_{-a} &= \alpha_{-1}A_{-1} + \alpha_{-2}A_{-2} + a\beta_{-1}A_{-1} + a\beta_{-2}A_{-2} \\ A_{-b} &= \alpha_{-1}A_{-1} + \alpha_{-2}A_{-2} - a\beta_{-1}A_{-1} - a\beta_{-2}A_{-2}, \end{aligned} \tag{57}$$

the above equations become

$$A_a^+ \Omega^{-1} A_{-a} \neq 0 \qquad A_b^+ \Omega^{-1} A_{-b} \neq 0. \tag{58}$$

In order to obtain the relations

$$\begin{aligned} A_a^+ \Omega^{-1} A_{-b} &= A_1^+ A_1 - aA_1^+ A_2 + a^* A_2^+ A_1 \\ &\quad - |a|^2 A_2^+ A_2 = 0 \\ A_b^+ \Omega^{-1} A_{-a} &= A_1^+ A_1 + aA_1^+ A_2 - a^* A_2^+ A_1 \\ &\quad - |a|^2 A_2^+ A_2 = 0 \end{aligned} \tag{59}$$

we need only to put

$$|a| = \sqrt{\frac{A_1^+ A_1}{A_2^+ A_2}}, \qquad \angle a = -\angle A_1^+ A_2. \tag{60}$$

Since $a \neq 0$, A_a, A_b, A_{-a} and A_{-b} thus defined are independent to each other and they satisfy the orthogonality theorems. In case $|\lambda_1| = 1$, similarly the modes can be introduced so as to secure the orthogonality. The generalization of the above discussion to the case of multiple degeneracy is not difficult.

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[8] Eq. (41) requires transmission lines without a cutoff effect. The effect of cutoff would be

$$\theta_p = \theta_1 + \theta_2 + \Delta\theta.$$

In this case, (51) becomes

$$G \doteq \cosh^2 n\delta' + \left(\frac{\Delta\theta}{\delta^9}\right)^2 \sinh^2 n\delta', \quad \text{where} \quad \delta' = \sqrt{\delta^2 - (\Delta\theta)^2}.$$